

INVERSE TRIGONOMETRIC FUNCTIONS

BASIC CONCEPTS

INVERSE CIRCULAR FUNCTIONS

Function	Domain	Range
1. $y = \sin^{-1} x$ iff $x = \sin y$	$-1 \leq x \leq 1$,	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2. $y = \cos^{-1} x$ iff $x = \cos y$	$-1 \leq x \leq 1$	$[0, \pi]$
3. $y = \tan^{-1} x$ iff $x = \tan y$	$-\infty < x < \infty$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4. $y = \cot^{-1} x$ iff $x = \cot y$	$-\infty < x < \infty$	$[0, \pi]$
5. $y = \operatorname{cosec}^{-1} x$ iff $x = \operatorname{cosec} y$	$(-\infty, -1] \cup [1, \infty]$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
6. $y = \sec^{-1} x$ iff $x = \sec y$	$(-\infty, -1] \cup [1, \infty]$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Note...

(i) $\sin^{-1} x$ & $\tan^{-1} x$ are increasing functions in their domain.

(ii) $\cos^{-1} x$ & $\cot^{-1} x$ are decreasing functions in over domain.

PROPERTY – I

(i) $\sin^{-1} x + \cos^{-1} x = \pi/2$, for all $x \in [-1, 1]$

Sol. Let, $\sin^{-1} x = \theta$... (i)

then, $\theta \in [-\pi/2, \pi/2]$ $[\because x \in [-1, 1]]$

$\Rightarrow -\pi/2 \leq \theta \leq \pi/2$

$\Rightarrow -\pi/2 \leq -\theta \leq \pi/2$

$\Rightarrow 0 \leq \frac{\pi}{2} - \theta \leq \pi$

$\Rightarrow \frac{\pi}{2} - \theta \in [0, \pi]$

Now, $\sin^{-1} x = \theta$

$$\Rightarrow x = \sin \theta$$

$$\Rightarrow x = \cos \left(\frac{\pi}{2} - \theta \right)$$

$$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - \theta$$

$\{\because x \in [-1, 1] \text{ and } (\pi/2 - \theta) \in [0, \pi]\}$

$$\Rightarrow \theta + \cos^{-1} x = \pi/2 \quad \dots \text{(ii)}$$

from (i) and (ii), we get

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

(ii) $\tan^{-1} x + \cot^{-1} x = \pi/2$, for all $x \in \mathbb{R}$

Sol. Let, $\tan^{-1} x = \theta$... (i)

then, $\theta \in (-\pi/2, \pi/2)$ $\{\because x \in \mathbb{R}\}$

$$\Rightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{2} < -\theta < \frac{\pi}{2}$$

$$\Rightarrow 0 < \frac{\pi}{2} - \theta < \pi$$

$$\Rightarrow \left(\frac{\pi}{2} - \theta\right) \in (0, \pi)$$

Now, $\tan^{-1} x = \theta$

$$\Rightarrow x = \tan \theta$$

$$\Rightarrow x = \cot(\pi/2 - \theta)$$

$$\Rightarrow \cot^{-1} x = \frac{\pi}{2} - \theta \quad \{\because \pi/2 - \theta \in (0, \pi)\}$$

$$\Rightarrow \theta + \cot^{-1} x = \frac{\pi}{2} \quad \dots \text{(ii)}$$

from (i) and (ii), we get

$$\tan^{-1} x + \cot^{-1} x = \pi/2$$

(iii) $\sec^{-1} x + \cosec^{-1} x = \frac{\pi}{2}$, for all $x \in (-\infty, -1] \cup [1, \infty)$

Sol. Let, $\sec^{-1} x = \theta$... (i)

then, $\theta \in [0, \pi] - \{\pi/2\}$ $\{\because x \in (-\infty, -1] \cup [1, \infty)\}$

$$\Rightarrow 0 \leq \theta \leq \pi, \theta \neq \pi/2$$

$$\Rightarrow -\pi \leq -\theta \leq 0, \theta \neq \pi/2$$

$$\Rightarrow -\frac{\pi}{2} \leq \frac{\pi}{2} - \theta \leq \frac{\pi}{2}, \frac{\pi}{2} - \theta \neq 0$$

$$\Rightarrow \left(\frac{\pi}{2} - \theta\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \frac{\pi}{2} - \theta \neq 0$$

Now, $\sec^{-1} x = \theta$

$$\Rightarrow x = \sec \theta$$

$$\Rightarrow x = \cosec(\pi/2 - \theta)$$

$$\Rightarrow \cosec^{-1} x = \pi/2 - \theta$$

$$\left\{ \because \left(\frac{\pi}{2} - \theta\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \frac{\pi}{2} - \theta \neq 0 \right\}$$

$$\Rightarrow \theta + \cosec^{-1} x = \pi/2 \quad \dots \text{(ii)}$$

from (i) and (ii); we get

$$\sec^{-1} x + \cosec^{-1} x = \pi/2$$

PROPERTY – II

(i) $\sin^{-1} \left(\frac{1}{x}\right) = \cosec^{-1} x$, for all $x \in (-\infty, 1] \cup [1, \infty)$

Sol. Let, $\cosec^{-1} x = \theta$... (i)
then, $x = \cosec \theta$

$$\Rightarrow \frac{1}{x} = \sin \theta$$

$$\{\because x \in (-\infty, -1] \cup [1, \infty) \Rightarrow \frac{1}{x} \in [-1, 1] \setminus \{0\}\}$$

$$\cosec^{-1} x = \theta \Rightarrow \theta \in [-\pi/2, \pi/2] - \{0\}$$

$$\Rightarrow \theta = \sin^{-1} \left(\frac{1}{x}\right) \quad \dots \text{(ii)}$$

from (i) and (ii); we get

$$\sin^{-1} \left(\frac{1}{x}\right) = \cosec^{-1} x$$

(ii) $\cos^{-1} \left(\frac{1}{x}\right) = \sec^{-1} x$, for all $x \in (-\infty, 1] \cup [1, \infty)$

Sol. Let, $\sec^{-1} x = \theta$... (i)

then, $x \in (-\infty, 1] \cup [1, \infty)$ and $\theta \in [0, \pi] - \{\pi/2\}$

Now, $\sec^{-1} x = \theta$

$$\Rightarrow x = \sec \theta$$

$$\Rightarrow \frac{1}{x} = \cos \theta$$

$$\Rightarrow \theta = \cos^{-1} \left(\frac{1}{x}\right) \quad \dots \text{(ii)}$$

$\left\{ \because x \in (-\infty, -1] \cup [1, \infty) \right.$

$$\left. \Rightarrow \frac{1}{x} \in [-1, 1] - \{0\} \text{ and } \theta \in [0, \pi] \right.$$

from (i) & (ii), we get

$$\cos^{-1} \left(\frac{1}{x}\right) = \sec^{-1} (x)$$

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$$(iii) \tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x, & \text{for } x > 0 \\ -\pi + \cot^{-1}x, & \text{for } x < 0 \end{cases}$$

Sol. Let $\cot^{-1}x = \theta$. Then $x \in \mathbb{R}$, $x \neq 0$ and $\theta \in [0, \pi]$... (i)

Now two cases arises :

Case I : When $x > 0$

In this case, $\theta \in (0, \pi/2)$

$$\therefore \cot^{-1}x = \theta$$

$$\Rightarrow x = \cot \theta$$

$$\Rightarrow \frac{1}{x} = \tan \theta$$

$$\theta = \tan^{-1}\left(\frac{1}{x}\right) \quad \dots (ii)$$

from (i) and (ii), we get $\{\because \theta \in (0, \pi/2)\}$

$$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}x, \text{ for all } x > 0.$$

Case II : When $x < 0$

In this case $\theta \in (\pi/2, \pi)$ $\{\because x = \cot \theta < 0\}$

$$\text{Now, } \frac{\pi}{2} < \theta < \pi$$

$$\Rightarrow -\frac{\pi}{2} < \theta - \pi < 0$$

$$\Rightarrow \theta - \pi \in (-\pi/2, 0)$$

$$\therefore \cot^{-1}x = \theta$$

$$\Rightarrow x = \cot \theta$$

$$\Rightarrow \frac{1}{x} = \tan \theta$$

$$\Rightarrow \frac{1}{x} = -\tan(\pi - \theta)$$

$$\Rightarrow \frac{1}{x} = \tan(\theta - \pi) \quad \{\because \tan(\pi - \theta) = -\tan \theta\}$$

$$\Rightarrow \theta - \pi = \tan^{-1}\left(\frac{1}{x}\right) \quad \{\because \theta - \pi \in (-\pi/2, 0)\}$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{x}\right) = -\pi + \theta \quad \dots (iii)$$

from (i) and (iii), we get

$$\tan^{-1}\left(\frac{1}{x}\right) = -\pi + \cot^{-1}x, \text{ if } x < 0$$

Hence,

$$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x, & \text{for } x > 0 \\ -\pi + \cot^{-1}x, & \text{for } x < 0 \end{cases}$$

PROPERTY – III

- (i) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$, for all $x \in [-1, 1]$
- (ii) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, for all $x \in (-\infty, -1] \cup [1, \infty)$
- (iii) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, for all $x \in \mathbb{R}$
- (iv) $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$
- (v) $\tan^{-1}(-x) = -\tan^{-1}x$, for all $x \in \mathbb{R}$
- (vi) $\cosec^{-1}(-x) = -\cosec^{-1}x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

Sol. (ii) Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$

let $\cos^{-1}(-x) = \theta \quad \dots (i)$

then, $-x = \cos \theta$

$$\Rightarrow x = -\cos \theta$$

$$\Rightarrow x = \cos(\pi - \theta)$$

$\{\because x \in [-1, 1] \text{ and } \pi - \theta \in [0, \pi] \text{ for all } \theta \in [0, \pi]\}$

$$\cos^{-1}x = \pi - \theta$$

$$\Rightarrow \theta = \pi - \cos^{-1}x \quad \dots (ii)$$

from (i) and (ii), we get

$$\cos^{-1}(-x) = \pi - \cos^{-1}x$$

Similarly, we can prove other results.

(i) Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$

$$\text{let } \sin^{-1}(-x) = \theta$$

then, $-x = \sin \theta \quad \dots (i)$

$$\Rightarrow x = -\sin \theta$$

$$\Rightarrow x = \sin(-\theta)$$

$$\Rightarrow -\theta = \sin^{-1}x$$

$\{\because x \in [-1, 1] \text{ and } -\theta \in [-\pi/2, \pi/2] \text{ for all } \theta \in [-\pi/2, \pi/2]\}$

$$\Rightarrow \theta = -\sin^{-1}x \quad \dots (ii)$$

from (i) and (ii), we get

$$\sin^{-1}(-x) = -\sin^{-1}(x)$$

PROPERTY – IV

- (i) $\sin(\sin^{-1}x) = x$, for all $x \in [-1, 1]$
- (ii) $\cos(\cos^{-1}x) = x$, for all $x \in [-1, 1]$
- (iii) $\tan(\tan^{-1}x) = x$, for all $x \in \mathbb{R}$
- (iv) $\text{cosec}(\text{cosec}^{-1}x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$
- (v) $\sec(\sec^{-1}x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$
- (vi) $\cot(\cot^{-1}x) = x$, for all $x \in \mathbb{R}$

Sol. We know that, if $f : A \rightarrow B$ is a bijection, then $f^{-1} : B \rightarrow A$ exists such that $f \circ f^{-1}(y) = f(f^{-1}(y)) = y$ for all $y \in B$.

Clearly, all these results are direct consequences of this property.

Aliter : Let $\theta \in [-\pi/2, \pi/2]$ and $x \in [-1, 1]$ such that $\sin \theta = x$

then, $\theta = \sin^{-1}x$

$$\therefore x = \sin \theta = \sin(\sin^{-1}x)$$

Hence, $\sin(\sin^{-1}x) = x$ for all $x \in [-1, 1]$

Similarly, we can prove other results.

Remark : It should be noted that,

$\sin^{-1}(\sin \theta) \neq \theta$, if $\theta \notin [-\pi/2, \pi/2]$. Infact, we have

$$\sin^{-1}(\sin \theta) = \begin{cases} -\pi - \theta, & \text{if } \theta \in [-3\pi/2, -\pi/2] \\ 0, & \text{if } \theta \in [-\pi/2, \pi/2] \\ \pi - \theta, & \text{if } \theta \in [\pi/2, 3\pi/2] \\ -2\pi + \theta, & \text{if } \theta \in [3\pi/2, 5\pi/2] \end{cases} \text{ and so on.}$$

Similarly,

$$\cos^{-1}(\cos \theta) = \begin{cases} -\theta, & \text{if } \theta \in [-\pi, 0] \\ \theta, & \text{if } \theta \in [0, \pi] \\ 2\pi - \theta, & \text{if } \theta \in [\pi, 2\pi] \\ -2\pi + \theta, & \text{if } \theta \in [2\pi, 3\pi] \end{cases} \text{ and so on.}$$

$$\tan^{-1}(\tan \theta) = \begin{cases} -\pi - \theta, & \text{if } \theta \in [-3\pi/2, -\pi/2] \\ \theta, & \text{if } \theta \in [-\pi/2, \pi/2] \\ \theta - \pi, & \text{if } \theta \in [\pi/2, 3\pi/2] \\ \theta - 2\pi, & \text{if } \theta \in [3\pi/2, 5\pi/2] \end{cases} \text{ and so on.}$$

PROPERTY – V

- (i) Sketch the graph for $y = \sin^{-1}(\sin x)$

Sol. As, $y = \sin^{-1}(\sin x)$ is periodic with period 2π .

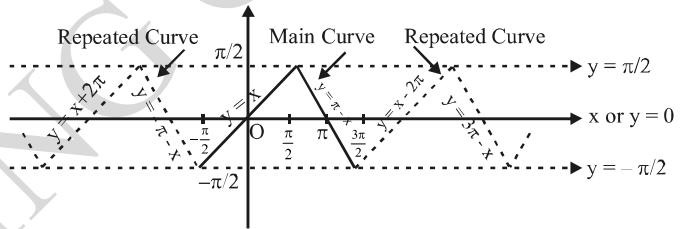
\therefore to draw this graph we should draw the graph for one interval of length 2π and repeat for entire values of x .

As we know,

$$\sin^{-1}(\sin x) = \begin{cases} x; & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ (\pi - x); & -\frac{\pi}{2} \leq \pi - x < \frac{\pi}{2} \left(\text{i.e., } \frac{\pi}{2} \leq x \leq \frac{3\pi}{2} \right) \end{cases}$$

$$\text{or } \sin^{-1}(\sin x) = \begin{cases} x, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \frac{3\pi}{2}, \end{cases}$$

which is defined for the interval of length 2π , plotted as ;



Thus, the graph for $y = \sin^{-1}(\sin x)$, is a straight line up and a straight line down with slopes 1 and -1 respectively lying

$$\text{between } \left[-\frac{\pi}{2}, \frac{\pi}{2} \right].$$



Students are advised to learn the definition of $\sin^{-1}(\sin x)$ as,

$$y = \sin^{-1}(\sin x) = \begin{cases} x + 2\pi; & -\frac{5\pi}{2} \leq x \leq -\frac{3\pi}{2} \\ -\pi - x; & -\frac{3\pi}{2} \leq x \leq -\frac{\pi}{2} \\ x; & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ \pi - x; & \frac{\pi}{2} \leq x \leq \frac{3\pi}{2} \\ x - 2\pi; & \frac{3\pi}{2} \leq x \leq \frac{5\pi}{2} \end{cases} \dots \text{and so on}$$

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(ii) Sketch the graph for $y = \cos^{-1}(\cos x)$.

Sol. As, $y = \cos^{-1}(\cos x)$ is periodic with period 2π .

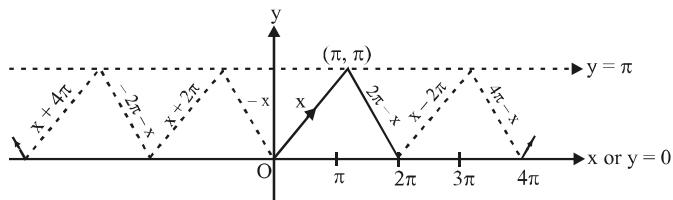
\therefore to draw this graph we should draw the graph for one interval of length 2π and repeat for entire values of x of length 2π .

As we know;

$$\cos^{-1}(\cos x) = \begin{cases} x; & 0 \leq x \leq \pi \\ 2\pi - x; & 0 \leq 2\pi - x \leq \pi, \end{cases}$$

or $\cos^{-1}(\cos x) = \begin{cases} x; & 0 \leq x \leq \pi \\ 2\pi - x; & \pi \leq x \leq 2\pi, \end{cases}$

Thus, it has been defined for $0 < x < 2\pi$ that has length 2π .
So, its graph could be plotted as;



Thus, the curve $y = \cos^{-1}(\cos x)$.

(iii) Sketch the graph for $y = \tan^{-1}(\tan x)$.

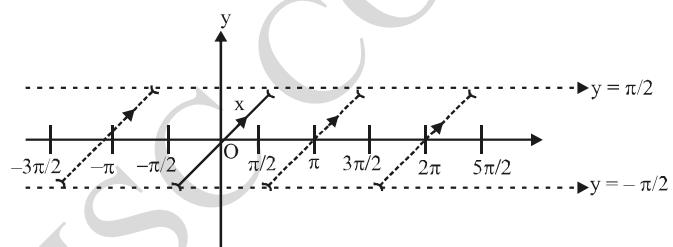
Sol. As $y = \tan^{-1}(\tan x)$ is periodic with period π .

\therefore to draw this graph we should draw the graph for one interval of length π and repeat for entire values of x .

As we know; $\tan^{-1}(\tan x) = \left\{ x; -\frac{\pi}{2} < x < \frac{\pi}{2} \right\}$

Thus, it has been defined for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ that has length π .

So, its graph could be plotted as;



Thus, the curve for $y = \tan^{-1}(\tan x)$, where y is not defined

for $x \in (2n+1)\frac{\pi}{2}$.

FORMULAS

(i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}, xy < 1$

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}, xy > -1$

(iii) $2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}, |x| < 1$

(iv) $2 \tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2}, |x| \leq 1$

(v) $2 \tan^{-1} x = \cos^{-1} \frac{1-x^2}{1+x^2}, x \geq 0$

(vi) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}(x \sqrt{1-y^2} + y \sqrt{1-x^2})$

(vii) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}(x \sqrt{1-y^2} - y \sqrt{1-x^2})$

(viii) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}(xy - \sqrt{1-x^2} \sqrt{1-y^2})$

(ix) $\cos^{-1} x - \cos^{-1} y = \cos^{-1}(xy + \sqrt{1-x^2} \sqrt{1-y^2})$

(x) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}$

$$\left[\frac{x+y+z-xyz}{1-xy-yz-zx} \right] \text{ if, } x > 0, y > 0, z > 0 \&$$

$$xy + yz + zx < 1$$

Note:

(i) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$ then $x + y + z = xyz$

(ii) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$ then $xy + yz + zx = 1$

REMEMBER THAT:

(i) $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2} \Rightarrow x = y = z = 1$

(ii) $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi \Rightarrow x = y = z = -1$

(iii) $\tan^{-1} 1 + \tan^{-1} 2 + 2 \tan^{-1} 3 =$

$$\tan^{-1} 1 + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{2}$$